HYDRODYNAMICS OF A SYSTEM OF BUBBLES IN A LOW-VISCOSITY LIQUID

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This paper deals with the effect of the gas content and the shape of the space occupied by a system of bubbles on the rate at which the bubbles rise in an infinite medium and in a vertical cylindrical column. Deformations of the system favorable from the energy point of view are considered for an assumed homogeneous and isotropic distribution of bubbles in the system.

A theoretical description of the motion of gas bubble systems in liquids is necessary for the study of bubbling processes. This problem has been frequently considered for low Reynolds numbers ($Re \ll 1$) on the basis of the so-called cell model. In [1] an analogous model was used to describe the motion of a system of medium-size bubbles ($Re < 300$). It was assumed that each bubble is at any instant of time in the center of an imaginary spherical cell of liquid the radius of which equals the average distance between the centers of bubbles in the system. The normal component of the velocity of the liquid and the surface of the cell is zero. The former assumption is equivalent to the assumption of homogeneity and isotropy of the system, but the latter is by no means physically convincing. In the present paper we calculate the rising velocity of a system of medium-size bubbles in a liquid with low gas content; these results differ even qualitatively from those obtained on the basis of the cell model. This apparently suggests that the cell model is unsatisfactory, at least in the case of a liquid with low gas content.

1. Formulation of the problem. It is assumed that all bubbles have the same radius R and the rising velocity of the bubbles is independent of time. Levich [2] showed that the motion of a single bubble with radius $R < 0.05$ cm in a liquid with a kinematic viscosity $\nu \sim 0.01$ cm^2/sec is approximately the same as that of an ideal liquid provided that surface-active substances are absent. Therefore as a first approximation for a system of medium-size gas bubbles for which a (average center distance) is much larger than the bubble dimensions we may assume an irrotational flow of an ideal liquid about a system of N spheres, the centers of which are located at points r_i with coordinates r_i^α ($\alpha = 1, 2, 3, i = 1, \ldots, N$).

In order to determine the velocity of the potential flow $v = \nabla\Phi$ it is necessary to solve the Laplace equation

$$\Delta\Phi = 0 \text{ in the region, except for } r_i' < R, \quad r_i' = |\mathbf{r}_i'|, \qquad \mathbf{r}_i' = \mathbf{r} - \mathbf{r}_i, \tag{1.1}$$

with boundary conditions expressing the fact that the radial component of the flow velocity vanishes on the surface of the i-th sphere in a coordinate system moving with the i-th sphere

$$\xi_i^\alpha \left(\frac{\partial}{\partial r^\alpha}\Phi - u_i^\alpha\right) = 0 \quad \text{as} \quad \xi_i = \frac{r_i'}{R} = 1 \quad (i = 1, \ldots, N), \tag{1.2}$$

where u_i^α is the velocity component of the i-th bubble. In (1.2) and in what follows we take the summation with respect to the repeated Greek indices. We seek Φ in a coordinate system in which the liquid is at rest as $r \to \infty$. It is obvious that this can be achieved by imposing the condition

$$\Phi \to 0 \quad \text{as } r \to \infty. \tag{1.3}$$

2. Velocity field potential. We seek the solution of Eq. (1.1), which satisfies the boundary conditions (1.2) and (1.3), in the following form:

$$\Phi = -\frac{R^3}{2}\sum_{i=1}^N V_i^\alpha \frac{r_i'^\alpha}{r_i'^3}, \qquad V_i^\alpha = u_i^\alpha - \frac{R^3}{2}\sum_{j=1}^N V_j^\beta A_{ji}^{\beta\alpha}. \tag{2.1}$$

Using the well-known Taylor expansion

$$\frac{r_i'^\alpha}{r_i'^3} = -\frac{\partial}{\partial r^\alpha}\left(S_{ik}\frac{1}{r_{ik}}\right) = \frac{\partial}{\partial r_i^\alpha}\left(S_{ik}\frac{1}{r_{ik}}\right) \tag{2.2}$$

written in terms of the displacement operator

$$S_{ik} = \exp\left(-r_k'^\alpha \frac{\partial}{\partial r_i^\alpha}\right) \qquad \left(\begin{matrix} r_{ik} = |\mathbf{r}_{ik}| \\ \mathbf{r}_{ik} = \mathbf{r}_i - \mathbf{r}_k \end{matrix}\right), \tag{2.3}$$

we can represent the potential Φ in the vicinity of the k-th bubble in the following form:

$$\Phi = -\frac{R^3}{2}\frac{u_k^\alpha r_k'^\alpha}{r_k'^3} + \frac{R^6}{4}\sum_{i=1}^N V_i^\alpha A_{ik}^{\alpha\beta}\frac{r_k'^\beta}{r_k'^3} - \frac{R^3}{2}\sum_{\substack{i=1 \\ i\neq k}}^{N}{}' V_i^\alpha \frac{\partial}{\partial r_i^\alpha}\left(S_{ik}\frac{1}{r_{ik}}\right). \tag{2.4}$$

It follows that at $r_k' = R$

$$\xi_k^\alpha \frac{\partial\Phi}{\partial r^\alpha} = u_k^\alpha \xi_k^\alpha + \frac{R^4}{4}\sum_{i=1}^N \frac{\partial V_i^\alpha A_{ik}^{\alpha\beta}}{\partial r_k'}\xi_k^\beta - \frac{R^3}{2}\sum_{i=1}^N V_i^\alpha A_{ik}^{\alpha\beta}\xi_k^\beta - \frac{R^3}{2}\sum_{i=1}^N \frac{\partial V_i^\alpha}{\partial r_k'}\frac{\partial}{\partial r_i^\alpha}\left(S_{ik}\frac{1}{r_{ik}}\right) + \frac{R^3}{2}\sum_{i=1}^N V_i^\alpha \xi_k^\beta S_{ik}\Lambda_{ik}^{\alpha\beta}, \tag{2.5}$$

where the tensor $\Lambda_{ik}^{\alpha\beta}$ is determined as

$$\Lambda_{ii}^{\alpha\beta} \equiv 0, \qquad \Lambda_{ik}^{\alpha\beta} = \frac{3 r_{ik}^\alpha r_{ik}^\beta}{r_{ik}^5} - \frac{\delta_{\alpha\beta}}{r_{ik}^3}, \quad (i \neq k), \qquad \delta_{\alpha\beta} = \begin{cases} 1, (\alpha = \beta) \\ 0 \ (\alpha \neq \beta) \end{cases}. \tag{2.6}$$

The boundary condition (1.2) can be used to determine the tensor $A_{ik}^{\alpha\beta}$.

In order to satisfy (1.2) with an accuracy to terms of order $(R/a)^6$ inclusive, we must select $A_{ik}^{\alpha\beta}$ so that the condition

$$\frac{R}{2}\frac{\partial A_{ik}^{\alpha\beta}}{\partial r_k'} - A_{ik}^{\alpha\beta} + S_{ik}\Lambda_{ik}^{\alpha\beta} = 0 \tag{2.7}$$

is satisfied at $r_k' = R$.

It can be shown by direct verification that the following tensor satisfies this condition:

$$A_{ik}^{\alpha\beta} = 2\zeta_{ik}^{-2}\left[1-(1+\zeta_{ik})e^{-\zeta_{ik}}\right]\Lambda_{ik}^{\alpha\beta},$$
$$\left(\zeta_{ik} = R^2\frac{r_k^{'\gamma}}{r_k^{'3}}\frac{\partial}{\partial r_i^\gamma}\right). \tag{2.8}$$

With an accuracy to the terms of order $(R/a)^6$ inclusive, the potential and its derivatives on the surface of the k-th bubble can be expressed in the following form:

$$\Phi = -\frac{R}{2}u_k^\alpha\xi_k^\alpha + \frac{R^3}{2}\sum_{\substack{i=1\\i\neq k}}^N \frac{v_i^\alpha r_{ik}^\alpha}{r_{ik}^3} + \frac{R^4}{2}\sum_{i=1}^N v_i^\alpha B_{ik}^{\alpha\beta}\xi_k^\beta, \tag{2.9}$$

$$\frac{\partial\Phi}{\partial r^\alpha} = \frac{3}{2}u_k^\beta\xi_k^\beta\xi_k^\alpha - \frac{1}{2}u_k^\alpha + \frac{R^3}{2}\sum_{i=1}^N v_i^\beta C_{ik}^{\alpha\gamma}\Lambda_{ik}^{\beta\gamma}, \tag{2.10}$$

$$r_k^{'\beta}\frac{\partial^2\Phi}{\partial r^\alpha\partial r^\beta} = -\frac{9}{2}u_k^\beta\xi_k^\beta\xi_k^\alpha + \frac{3}{2}u_k^\alpha - \frac{R^3}{2}\sum_{i=1}^N v_i^\beta D_{ik}^{\alpha\gamma}\Lambda_{ik}^{\beta\gamma},$$

$$v_i^\alpha = u_i^\alpha - \frac{R^3}{2}\sum_{j=1}^N v_j^\beta\Lambda_{ji}^{\beta\alpha}, \tag{2.11}$$

$$B_{ik}^{\alpha\beta} = \frac{1}{2}A_{ik}^{\alpha\beta} - \zeta_{ik}^{-1}(e^{-\zeta_{ik}}-1)\Lambda_{ik}^{\alpha\beta},$$

$$C_{ik}^{\alpha\gamma} = \frac{\partial}{\partial r_k^{'\alpha}}\sum_{n=0}^{3}\frac{2n+3}{(n+2)!}\left(\frac{R}{r_k^{'}}\right)^{n+1} r_k^{'\gamma}\left(-r_k^{'\lambda}\frac{\partial}{\partial r_i^\lambda}\right)^n,$$

$$D_{ik}^{\alpha\gamma} = \frac{\partial}{\partial r_k^{'\alpha}}\sum_{n=0}^{3}\frac{2n+3}{(n+2)!}\left(\frac{R}{r_k^{'}}\right)^{(n+1)(n+3)} r_k^{'\gamma}\left(-r_k^{'\lambda}\frac{\partial}{\partial r_i^\lambda}\right)^n.$$

3. **Kinematic energy of the liquid and the forces acting on the sphere.** Knowing the velocity field potential, it is possible to calculate the kinetic energy of the liquid as an integral over the entire space occupied by the liquid with density ρ

$$T = \frac{\rho}{2}\int(\nabla\Phi)^2 d^3r, \quad\text{or}\quad T = -\frac{\rho}{2}\sum_{k=1}^N\int\Phi u_k^\alpha\xi_k^\alpha dS_k. \tag{3.1}$$

Here the integration is carried out over the surface of each sphere. The negative sign appears because the normal to the surface S_k, which bounds the volume of the liquid, acts in a direction opposite to ξ_k, i.e., the unit vector of the exterior normal to the surface of the k-th sphere.

Thus, we see that with an accuracy to the terms of order $(R/a)^6$

$$T = \frac{\pi\rho R^3}{3}\left(\sum_{k=1}^N u_k^2 - \frac{3R^3}{2}\sum_{i,\,k=1}^N v_i^\alpha\Lambda_{ik}^{\alpha\beta}u_k^\beta\right). \tag{3.2}$$

In his monograph [3] Lamb calculates with the same accuracy the kinetic energy of the liquid in which two spheres of any radius move normal to and along the center line. The results agree with Eq. (3.2).

The total energy of the entire system is the same as the kinetic energy of the liquid, which can be represented as an integral over the space occupied by the liquid or as the total energy of a system of interacting spheres with additional masses moving in vacuum.

It should be noted that these two interpretations of the energy of the system are analogous to the description of the energy of the system in terms of the energy of the field and on the basis of the theory of distant interaction in electrostatics.

The energy of a system of spheres can be interpreted as a sum of their kinetic energies and the energy of interaction. However, the division of energy into kinetic and potential energies is not unambiguous. It may be assumed that the first sum of (3.2) describes the kinetic energy of spheres moving at velocities $\mathbf{u}_k$ with additional masses equal to half the mass of the liquid in the volume of the bubble. In this case the interaction energy may be considered as a sum of dipole-dipole interactions between pairs of spheres traveling at velocities $\mathbf{u}_i$ and $\mathbf{u}_k$, where one has the same dipole moment as an isolated sphere traveling at the velocity $\mathbf{u}_k$ in an unbounded medium, while the other travels as an isolated sphere with an effective velocity $\mathbf{v}_i$. We could include in the kinetic energy the terms of the second sum of (3.2) which contain squares of the velocities. In the case of such an interpretation the additional masses depend on the location of all bubbles in the system.

Using a variational method Breakwell [4] showed that a system of spheres in an ideal liquid is a Lagrangian system. Therefore, if an external force $\mathbf{F}_k^*$ is acting on the k-th bubble the system of equations describing the motion of the spheres must have the following form:

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}_k} - \frac{\partial T}{\partial r_k} = \mathbf{F}_k^*. \tag{3.3}$$

In order to find the force $\mathbf{F}_k$ which acts on the k-th bubble and which is produced as the result of the hydrodynamic interaction of bubbles traveling at specified velocities in a flow of ideal liquid, we assume that the external force $\mathbf{F}_k^*$ exactly balances the force $\mathbf{F}_k$. It follows that

$$F_k^\alpha = \frac{\partial T}{\partial r_k^\alpha} - \sum_{n=1}^N u_n^\beta\frac{\partial}{\partial r_n^\beta}\frac{\partial T}{\partial u_k^\alpha}. \tag{3.4}$$

Direct calculations show that this result is correct. We denote by $\Phi_0 = \Phi - u_k^\alpha r^\alpha$ the velocity field potential in a system moving with the k-th bubble. Using the Bernoulli theorem, we can write

$$\mathbf{F}_k = \frac{\rho}{2}\int\left[(\nabla\Phi_0)^2 + \frac{\partial\Phi_0}{\partial t}\right]\xi_k dS_k. \tag{3.5}$$

In the coordinate system moving with the k-th bubble the total energy of the liquid T_0 is

$$T_0 = \frac{\rho}{2}\int(\nabla\Phi_0)^2 d^3r = T + \rho\sum_{n=1}^N\int\Phi u_k^\alpha\xi_n^\alpha dS_n + \text{const} =$$
$$= -T - \rho\sum_{n=1}^N\int\Phi u_{nk}^\alpha\xi_n^\alpha dS_n + \text{const}\quad (u_{nk}^\alpha = u_n^\alpha - u_k^\alpha). \tag{3.6}$$

The terms which are independent of the coordinates of the bubble centers are contained in "const." From Eq. (3.6) it follows that

$$\frac{\partial T}{\partial r_k^\alpha} = -\frac{\partial T_0}{\partial r_k^\alpha} - \rho\sum_{n=1}^N\frac{\partial}{\partial r_k^\alpha}\int\Phi u_{nk}^\beta\xi_n^\beta dS_n =$$
$$= \frac{\rho}{2}\int(\nabla\Phi_0)^2\xi_k^\alpha dS_k - \rho\int\frac{\partial\Phi_0}{\partial r^\beta}\frac{\partial^2\Phi_0}{\partial r_k^\alpha\partial r^\beta}d^3r -$$
$$-\frac{\rho}{2}\sum_{n=1}^N\int\frac{\partial\Phi}{\partial r_k^\alpha}u_{nk}^\beta\xi_k^\beta dS_n = \frac{\rho}{2}\int(\nabla\Phi_0)^2\xi_k^\alpha dS_k. \tag{3.7}$$

Then, using (3.2) we can show that

$$\frac{\rho}{2}\int \frac{\partial \Phi_0}{\partial t}\xi_k^\alpha dS_k + \sum_{n=1}^{N} u_n^\beta \frac{\partial}{\partial r_n^\beta}\frac{\partial T}{\partial u_k^\alpha} =$$

$$= \frac{\rho}{2}\sum_{n=1}^{N} u_{nk}^\beta \frac{\partial}{\partial r_n^\beta}\int \Phi \xi_\alpha^k dS_k - \pi\rho R^6 \sum_{n,m=1}^{N} u_n^\beta \frac{\partial}{\partial r_n^\beta} v_m^\gamma \Lambda_{mk}^{\gamma\alpha} =$$

$$= -\pi\rho R^6 \sum_{n,m=1}^{N} u_k^\beta \frac{\partial}{\partial r_n^\beta} v_m^\gamma \Lambda_{mk}^{\gamma\alpha} = 0. \tag{3.8}$$

Equations (3.7) and (3.8) confirm the relation (3.4), which permits the calculation of $\mathbf{F}_k$ from the known kinematic energy of the system. Thus, we arrive at the following result:

$$F_k^\alpha = \pi\rho R^6 \left(\sum_{i=1}^{N} v_i^\beta \frac{\partial \Lambda_{ik}^{\beta\gamma}}{\partial r_{ik}^\alpha} v_k{}^\gamma + \sum_{i,j=1}^{N} u_i^\beta \frac{\partial v_j^\gamma \Lambda_{jk}^{\gamma\alpha}}{\partial r_i{}^\beta} \right). \tag{3.9}$$

The first term of (3.9) expanded in series in powers of $(R/a)^3$ is

$$F_k^\alpha = \pi\rho R^6 \sum_{i=1}^{N} u_i^\beta \frac{\partial \Lambda_{ik}^{\beta\gamma}}{\partial r_{ik}^\alpha} u_i^\gamma . \tag{3.10}$$

This result is in agreement with the Bjerkness equation [5] for the force of interaction between two spheres of different radii moving at the same speed, and also with the expression for the force acting when two spheres travel in the direction of their centers with arbitrary velocities ([3], §138).

Let us now consider the motion of spheres with the same velocity under the condition that the center distance is commensurable with the sphere dimensions. Then, according to (3.9) the force of the pairwise interaction between bubbles moving at velocities $\mathbf{u}$ is determined by an equation analogous to the Bjerkness equation in which the actual velocities $\mathbf{u}$ are replaced by the effective velocities $\mathbf{v}$. In this case

$$\mathbf{F}_k = \partial T / \partial \mathbf{r}_k . \tag{3.11}$$

This type of the interrelation between the force acting on the bubble and the energy of the system in the case when the bubbles move at the same velocities suggests the conclusion that the system tends to a condition corresponding to a maximum of the total energy T calculated in the coordinate system in which the liquid is at rest at infinity.

In a low-viscosity liquid the k-th sphere is acted upon, in addition to force $\mathbf{F}_k$, also by the frictional force $\mathbf{Q}_k$ which can be calculated via the rate of energy dissipation

$$\mathbf{Q}_k = \frac{\mathbf{v}_k}{v_k^2}\frac{d}{dt} T_k , \tag{3.12}$$

where the rate of energy dissipation during flow past the k-th sphere is determined by an integral over the surface of the k-th sphere:

$$-\frac{d}{dt} T_k = -2\mu \int \xi_k^\alpha \frac{\partial \Phi_0}{\partial r^\beta}\frac{\partial^2 \Phi_0}{\partial r^\alpha \partial r^\beta} dS_k , \tag{3.13}$$

where μ is the dynamic viscosity of the liquid.

From (2.10) and (2.11) the result can be written in the following form:

$$-\frac{d}{dt} T_k = 12\pi\mu R v_k^2 . \tag{3.14}$$

This gives

$$\mathbf{Q}_k = -12\pi\mu R \mathbf{v}_k . \tag{3.15}$$

Thus, a bubble traveling at velocity $\mathbf{u}_k$ relative to the liquid at rest at infinity experiences in the presence of a system of bubbles the same resistance as an isolated bubble moving relative to the medium with an effective velocity $\mathbf{v}_k$.

The motion of the bubbles is assumed to be steady. Consequently, the sum of the forces (including the buoyancy force) acting on the bubble must be zero,

$$\mathbf{F}_k + \mathbf{Q}_k - \frac{4\pi}{3}\rho R^3 \mathbf{g} = 0 , \tag{3.16}$$

where $\mathbf{g}$ is the acceleration due to gravity.

This equation permits the calculation of the velocity of a rising bubble as a function of its size and the location of the other bubbles.

4. The Lorentz method. Lorentz's method [6] can be used to calculate the sums contained in the expression for the effective velocity provided that the following additional assumptions are made: a) the velocities of all the rising bubbles in the system are equal in magnitude and direction; b) the average distance between bubble centers is constant for the entire system, i.e., the average concentration of bubbles in the system $c = 4/3\,\pi R^3/a^3$ is constant, and c) the system has the shape of an ellipsoid or a shape representing a limiting case of an ellipsoid (a sphere, a thin plate, or a long circular cylinder).

If the centers of the bubbles are at the corners of a cubical system, then

$$R^3 \sum_{i=1}^{N} u_i^\alpha \Lambda_{ik}^{\alpha\beta} = (1 - 3n_z)\, c u_k^\beta , \tag{4.1}$$

where n_c is the coefficient of depolarization of the ellipsoid (it is assumed that the z axis coincides with the vector $\mathbf{u}_k$). It is known that for a spheroid with semiaxes $l_x = l_y < l_z$ elongated along the z axis

$$n_z = \frac{1-\varepsilon^2}{2\varepsilon^3}\left(\ln\frac{1+\varepsilon}{1-\varepsilon} - 2\varepsilon\right) \leqslant \frac{1}{3} \qquad \left(\varepsilon^2 = 1 - \frac{l_x^2}{l_z^2}\right), \tag{4.2}$$

and for a spheroid with semiaxes $l_x = l_y > l_z$ compressed in the direction of the z axis

$$n_z = \frac{1+\varepsilon^2}{\varepsilon^3}(\varepsilon - \operatorname{arc\,tg} \varepsilon) \geqslant \frac{1}{3} \qquad \left(\varepsilon^2 = \frac{l_x^2}{l_z^2} - 1\right). \tag{4.3}$$

Kirkwood and Ivon (see, for example, [7]) showed that the result (4.1) still applies for a geometrically uniform and isotropic distribution of dipoles as the first term of a series expansion in powers of the concentration.

5. System of bubbles in an unbounded liquid. If the conditions listed in the previous section are satisfied, then the force of hydrodynamic interaction of the bubbles $\mathbf{F}_k$ expressed by Eq. (3.9) is nonzero for any bubble with the exception of those near the surface of the system. For bubbles contained in the body of the system the dissipation force Q_k is balanced by the buoyancy force with the result that the effective speed of a bubble v_k is the same as the velocity of an isolated rising bubble u_0.

The relation (3.16) and Lorentz's method enable the velocity of the k-th rising bubble belonging to the system of bubbles to be expressed in terms of u_0:

$$\mathbf{u}_k = \mathbf{u}_0 \, [1 + {}^1/_2 \, (1 - 3n_z)c] . \qquad (5.1)$$

It should be noted that for medium-size bubbles the rising velocity depends on the radius in the following way [2]:

$$\mathbf{u}_0 = -{}^1/_9 \, R^2 \mathbf{g}/\nu . \qquad (5.2)$$

Thus, if the system of bubbles has the shape of a long circular cylinder ($n_z = 0$), then at the same concentration of bubbles it rises more rapidly than when the system is spherical ($n_z = 1/3$) or has the shape of a thin plate ($n_z = 1$).

For the rise of a system of bubbles shaped as a thin plate the results (5.1) can be explained as follows: as the bubbles rise at the velocity u_k their place is taken by liquid, which leads in the region occupied by bubbles to the formation of a descending flow having a velocity $-u_k c/(1 - c)$. Relative to the liquid moving at an average velocity of $-u_k c/(1 - c)$ the bubble rises with the velocity u_0, i.e., relative to liquid at rest at infinity it rises with the velocity $u_k = u_0(1 - c)$, which is in full agreement with the special case of Eq. (5.1).

It is obvious that the correctness of this special case is independent of the assumption that the bubbles are of medium size and have a spherical shape and that the intervals between individual bubbles are large compared with their dimensions. It is only important that the averaged motion of the liquid is one-dimensional and that the gas content in the region occupied by the bubbles is constant. In [1] calculations carried out for the cell model produced the following result:

$$u_k = u_0 \frac{(1 - 3c/4\pi)^2}{1 - (3c/4\pi)^{5/3}} . \qquad (5.3)$$

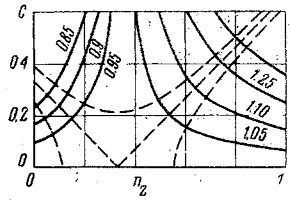
This result fails to take into account the dependence of the velocity with which the bubble system rises on its shape. In the region of small concentrations Eq. (5.3) is in qualitative agreement with (5.2) only if $n_z > 1/3$.

Since the effective velocity v_k was the same as the velocity of a single isolated bubble u_0, Eq. (3.12) shows that the rate of energy dissipation during the motion of a bubble in the system is the same as during the motion of a single bubble. Thus, from the point of view of the principle of minimum rate of energy dissipation [4], no conclusions can be drawn on the shape of the system.

The kinetic energy of the liquid dissipated during steady motion is completely compensated by the work of the external (buoyancy) force. For this reason the kinetic energy of the liquid, which is equal to its total energy, is independent of time and equals, with (3.2), (4.1), and (5.1) taken into account,

$$T = T_0 \, (1 - {}^1/_2 \eta c - {}^1/_2 \eta^2 c^2) , \qquad (5.4)$$
$$(T_0 = {}^1/_3 \rho \pi R^3 N u_0^2, \quad \eta = 1 - 3n_z) .$$

The figure shows as continuous lines in a plane with coordinates n_z, c the family of curves T/T_0 = const, while the family of orthogonal curves is indicated by broken lines.



The energy of the liquid reaches its maximum along the curve $c(n_z)$ in the region $n_z > 1/2$, which is determined by the relation

$$c(n_z) = \frac{1}{2(3n_z - 1)} . \qquad (5.5)$$

The system is deformed in the direction ∇T, i.e., along the corresponding broken curve passing through the initial state in the direction of increasing energy, as mentioned in discussing Eq. (3.11).

In this case, if the initial state of the system is in the region $c < 1/3 - n_z$ of the n_z, c plane, i.e., if the system is a low-concentration spheroid elongated in the direction of its axis, then the bubbles of which the system consists will tend to dissipate, so that the shape of the system will approach a sphere.

If, however, the initial condition is in the region $1/3 - n_z < c < 1/3$ (elongated spheroid with high bubble concentration), then the system begins to spread and deforms to a sphere, whereupon it tends to a condition with a higher concentration, deforming in this case to a slightly compressed spheroid. The deformation is complete when the system is in the state indicated in the figure by the point of intersection of the corresponding broken curve and the line $c(n_z)$ corresponding to the maximum-energy condition.

In the case of a condition in the region $n_z > 1/3$ at $c < c(n_z)$ the system tends towards the condition with a higher concentration, while its shape tends to a more compressed spheroid; at $c > c(n_z)$ the deformation of the system produces a less compressed spheroid and is accompanied by a certain reduction of bubble concentration.

These large variations in deformation are due to a competition in this process between the variation of the system velocity and the energy of the dipole-dipole interaction.

These conclusions concerning the behavior of the system hold if the distribution of bubbles during deformation of the system remains homogeneous and isotropic.

6. Introduction of the finite size of the region occupied by the liquid. Let us consider the motion of bubbles in a vertical column of liquid with finite dimensions. As long as the region occupied by bubbles is small in relation to the linear dimensions of the column, the rising velocity of the bubbles remains dependent on the shape of the system and the concentration in a way described by expression (5.1). However, if the dimensions of the region occupied by bubbles are commensurable with those of the region occupied by the liquid, the relation (5.1) cannot be used and Eq. (1.1) must be solved with the conditions (1.2) and (1.3) satisfied and with the additional condition that the normal component of the velocity of the liquid vanishes on the lateral surface of the column.

The solution is obtained most easily if the following requirements are met: 1) the distribution of bubbles in the system is homogeneous and isotropic; 2) the cross section of the column is a rectangle; and 3) the bubbles occupy a region between two cross sections of the column.

It is obvious that in order to satisfy the Laplace equation and all the boundary conditions it is necessary in expressing the potential to add to the field induced by the dipoles contained in the column a field of images consisting of dipoles arranged with the same density as the bubbles in the system outside the column in the region occupied by the infinite layer between the cross sections of the column.

Thus, it is necessary to solve the problem of the motion of bubbles occupying a region shaped as a thin plate and located in an infinite liquid medium. Regardless of the size of the column and the thickness of the layer occupied by the bubbles the absolute velocity of the rising bubbles which fill the space between two transverse cross sections of the column with uniform concentration is

$$\mathbf{u}_k = \mathbf{u}_0 (1 - c)\,. \qquad (6.1)$$

This result holds for a cylindrical column with any cross section, since it is essential only that the average motion of the liquid remain vertical, in which case (6.1) is a consequence of the discussion concerning Eq. (5.1). The total energy of the system is, according to (5.4),

$$T = T_0 (1 + c - 2c^2)\,. \qquad (6.2)$$

This function has a maximum at $c = 0.25$. The system should tend toward a condition with such a concentration only if the homogeneity and isotropy of the bubble distribution in the system is retained during the deformation process.

7. Continuous flow of bubbles. The case when the system of bubbles rises in a vertical tube containing a liquid with a continuous flow of bubbles being produced by a steady supply of air at its base requires special consideration. In such a case the average velocity of the liquid in any cross section of the column is zero and the absolute velocity of the rising bubbles (denoted by $\mathbf{u}_k$*) is $\mathbf{u}_0$ and remains independent of the concentration of bubbles in the system.

Davidson and Harrison [8] believe that for a continuous flow of air through the column the absolute velocity of the rising bubbles exceeds the velocity with which each individual bubble rises in the stationary liquid. In our notation their result has the following form:

$$\mathbf{u}_k{}^* = \mathbf{u}_0 + \frac{G}{A} = \frac{\mathbf{u}_0}{1-c} \qquad \left(\frac{G}{A} = c\mathbf{u}_k^*\right). \qquad (7.1)$$

Here G/A is the average velocity of gas in the free cross section of the column.

Equation (7.1) holds for sufficiently small periods of time when the bubbles have no time to reach the upper phase interface. In this case, in fact, the average velocity of the liquid in the cross section above the bubbles is G/A and, because the relative velocity is constant, Eq. (7.2) follows. However, if the continuous flow of bubbles in the column is of longer duration, then, because the average velocity of the liquid in any cross section of the column is zero, Eq. (7.1) is no longer correct.

Nicklin [9] showed the relationship between the absolute velocities of rising bubbles produced as a result of a continuous supply of air and that of bubbles occupying the space in the liquid between two cross sections

$$\mathbf{u}_k = \mathbf{u}_k{}^* - G\,/\,A, \qquad (7.2)$$

which, obviously, agrees with the results obtained in the present paper.

The total energy of the system during continuous flow of gas through the column is

$$T^* = T_0 (1 + 2c)\,. \qquad (7.3)$$

It follows from this equation that in the case of homogeneity and isotropy of the bubble distribution the system tends toward a condition with a higher concentration, which may result in a narrowing of the cross-sectional dimensions of the region occupied by bubbles in the direction of rise. Inhomogeneity of the gas content may result in a convection flow in the system.

Experiments [10, 11] clearly indicate an increase of the velocity of rising bubbles with increasing gas content. This can be caused by an increased coagulation of bubbles, which results in an increase of their average size and, consequently, in an increase of their velocity; by the convection flow of liquid in the column; or by the formation of "clusters," i.e., by accumulation of bubbles rising approximately as a single solid body with a velocity which is considerably higher than the velocity of an individual isolated bubble. In this case the liquid flow passes not round each individual bubble separately, but round an entire "cluster."

Attempts to explain the experimental results of [10, 11] on the basis of Eq. (7.1), which is irrelevant to the case in point, are useless.

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